AN APPLICATION OF SET THEORY TO MODEL THEORY*

BY

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ABSTRACT

We give a number of simple proofs of results in model theory using the set theoretical result of Lévy that $H(\mu)$ is a Σ-submodel of the Universe.

The results presented in this paper were noticed by the author during the early part of 1970, and are scattered throughout [6]. We collect these results together here because of their common nature and the common nature of their proofs. All the proofs are simple and use only a few facts, especially a result originally set down by Lévy [4]. In addition, we feel the first two results are of sufficient model theoretic interest in their own right.

In section 1, we list the facts we will need to get the results of section 2, 3 and 4. We will not attempt to derive them here, but recommend that the reader consult [5] or [6] if more than our brief explanations are desired. The original presentation of the now standard syntactical hierarcy of formulas of set theory is found in [4].

1. We will work with the infinitary language $\mathscr{L}_{\infty\omega}$, which allows the conjunction and disjunction of arbitrary sets of formulas, but only quantification over finitely many variables at a time. Some of our results will have immediate extensions to richer languages, and others, perhaps not so immediate extensions. Extending the results will essentially amount to extending the properties we mention below. In any event, we leave these extensions entirely to the reader.

We pause to introduce a little notation. By a language, we will more accurately mean the symbols of a language. We will usually denote languages by script $\mathscr{L}$.

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We will use script $\mathscr{M}$ and $\mathscr{N}$ as names for structures. M will denote the domain of $\mathscr{M}$. We denote the set of all k-tuples of elements of M, for $k \in \omega$, by $\vec{M}$, and individual k-tuples by $\vec{m}$, $\vec{p}$ etc. We write $\mathscr{M} \equiv_{\mathscr{L}_{\infty\omega}} \mathscr{N}$ to mean that $\mathscr{M}$ and $\mathscr{N}$ satisfy the same sentences of $\mathscr{L}_{\infty\omega}$. When the particular $\mathscr{L}$ involved is either clear or unimportant, we usually abbreviate with $\mathscr{M} \equiv_{\infty\omega} \mathscr{N}$. We write $\mathscr{M} \equiv^{\alpha}_{\infty\omega} \mathscr{N}$ to mean that $\mathscr{M}$ and $\mathscr{N}$ satisfy the same formulas of $\mathscr{L}_{\infty\omega}$ of quantifier rank no more than α, where α is an ordinal. For $\vec{m}$, $\vec{p} \in \vec{M}$, by $\vec{m} \sim_{\alpha} \vec{p}$ we mean that $(\mathscr{M}, \vec{m}) \equiv^{\alpha}_{\infty\omega} (\mathscr{M}, \vec{p})$ in the language with the appropriate number of symbols added.

We will denote cardinals with either μ or λ, and $H(\mu)$ will be the set of all sets of power hereditarily less than μ, i.e., whose transitive closures have power less than μ. $\beth_{\alpha}(\lambda)$ is defined inductively by $\beth_0(\lambda) = \lambda \cup \omega$, $\beth_{\alpha+1}(\lambda) = 2^{\beth_{\alpha}(\lambda)}$ and $\beth_{\delta}(\lambda) = \cup_{\beta<\delta} \beth_{\beta}(\lambda)$ if δ is a limit ordinal.

For our purposes the set of Δ_0-formulas of set theory contains the atomic formulas and is closed under the Boolean operations and bounded quantification. The set of Σ-formulas includes the Δ_0-formulas and is closed under existentia quantification, as well as conjunction, disjunction and bounded quantification. The Π-formulas are obtained in an analogous fashion using universal quantification. The negation of a Σ-formula is provably equivalent to a Π-formula, and vice versa. A class is said to be $\mathbf{\Sigma}$ ($\mathbf{\Pi}$) if it can be defined by a Σ-formula (Π-formula) using parameters. A class which is both $\mathbf{\Sigma}$ and $\mathbf{\Pi}$ is said to be $\mathbf{\Delta}$. In our setting, recursive definitions in terms of $\mathbf{\Delta}$ classes define $\mathbf{\Delta}$ classes.

Here are the properties we will use. We will not list the conditions for the "universe" under which these properties hold but will merely assure the reader that they hold in the cases we consider.

I) The relation '$\mathscr{M} \models \phi[\vec{m}]$', of the variables $\mathscr{M}$, ϕ, and $\vec{m}$, where $\mathscr{M}$ is a structure, ϕ is a formula of $\mathscr{L}_{\infty\omega}$, and $\vec{m} \in \vec{M}$ is (uniformly) $\mathbf{\Sigma}$.

This fact follows immediately from the inductive character of the definition of satisfaction, which even gives $\mathbf{\Delta}$-definability.

II) $\mathscr{M} \equiv_{\infty\omega} \mathscr{N}$ is a $\mathbf{\Sigma}$-relation.

Here we may use either the so-called "back-and forth property" of Karp, (cf. [1]) or a Chang canonical Scott sentence (cf. [2]). We note that $\equiv_{\infty\omega}$ is obviously $\mathbf{\Pi}$ as well.

III) If $\mathscr{M}$ and $\mathscr{N}$ are countable, then $\mathscr{M} \equiv_{\infty\omega} \mathscr{N}$ iff $\mathscr{M}$ and $\mathscr{N}$ are isomorphic.

This is well-known and needs no explanation here cf. [2].

All the above is exploited with the help of

IV) For each cardinal $\mu, H(\mu) < \Sigma V$, i.e., for any Σ-formula ϕ of set theory and any k-tuple $\vec{a}$ of elements of $H(\mu)$, $\langle H(\mu), \varepsilon \rangle \models \phi[\vec{a}]$ iff $\phi[\vec{a}]$.

This is, of course, the result of Lévy's to which we referred earlier.

2. Let S be a class of formulas of $\mathscr{L}_{\infty\omega}$. By $\mathscr{M}(S)\mathscr{N}$ we mean that for each sentence $\phi \in S$, if $\mathscr{M} \models \phi$, then $\mathscr{N} \models \phi$. In [2] Chang proved if $\mathscr{M}$ and $\mathscr{N}$ have cardinality less than μ, then $\mathscr{M}(S \cap \mathscr{L}_{\mu\omega})\mathscr{N}$ implies $\mathscr{M}(S)\mathscr{N}$, where S could be taken to be all formulas, all existential formulas, or all positive formulas of $\mathscr{L}_{\infty\omega}$. The above three classes are all of a very simple syntactical nature, and can be defined using only bounded quantifiers. We noticed that these three results are instances of a general result which has the added benefit of a uniform proof. Before beginning, we make a few simple observations.

Let $\mathscr{M}$ be a structure whose domain M has at most μ elements. Since there are at most 2^μ relations and functions on M, we may as well assume that the language $\mathscr{L}$ for $\mathscr{M}$ has no more than 2^μ symbols. In fact, we may assume that $M \in H(\mu^+)$, and that $\mathscr{L} \subseteq H(\mu^+)$, and even further that the mapping of the symbols of $\mathscr{L}$ to their interpretations in $\mathscr{M}$ is Δ on $H(\mu^+)$. We call such a structure an $H(\mu^+)$-structure, and as we have just observed, every structure whose domain has power at most μ is essentially isomorphic to an $H(\mu^+)$-structure. Hence, when we state a theorem for $H(\mu^+)$-structures, there is no loss of generality. In this setting, for a fixed $\mathscr{M}$, the class of $\phi \in \mathscr{L}_{\mu+\omega}$ such that $\mathscr{M} \models \phi$, is Δ on $H(\mu^+)$. We now state the result.

THEOREM 1. *Let $\mathscr{M}$ and $\mathscr{N}$ be $H(\mu^+)$-structures for a language $\mathscr{L}$ and suppose that S is a Σ-subclass of formulas of $\mathscr{L}_{\infty\omega}$ defined with parameters from $H(\mu^+)$. Then, if $\mathscr{M}(S \cap \mathscr{L}_{\mu\omega})\mathscr{N}$, $\mathscr{M}(S)\mathscr{N}$.*

PROOF. We think of S, $\mathscr{M}$, and $\mathscr{N}$ as their respective definition. Since S is Σ, it follows from Lévy's theorem that $S \cap H(\mu^+)$ is the same as S relativized to $H(\mu^+)$. Now we merely put down the following Π statement which expresses our hypothesis. $\langle H(\mu^+), \varepsilon \rangle \models$ "$\forall \phi[\phi$ is a sentence in $S \to [\mathscr{M} \models \phi \to \mathscr{N} \models \phi]]$". Using Lévy's theorem once again, the statement

(*) "$\forall \phi\, [\phi$ is a sentence in $S \to [\mathscr{M} \models \phi \to \mathscr{N} \models \phi]]$"

lifts to the "real" universe. This statement is not exactly what we want. $\mathscr{M}$ and $\mathscr{N}$ being classes and not merely elements, may be "blown up" in the real world interpretation. However, the domains remain the same and the original relations remain unchanged since these were elements of $H(\mu^+)$. The only possible alteration would be some additional symbols in the language and additional corresponding relations and functions. These additions clearly have no effect on the satisfaction of formulas of $\mathscr{L}_{\infty\omega}$, since the reduct of $\mathscr{M}$ as evaluated in the universe to $\mathscr{L}$, is the original structure $\mathscr{M}$, where, of course, by $\mathscr{L}$ we mean the original set of symbols and not the possibly inflated set we might get by using the definition of $\mathscr{L}$ in the universe. It is clear now that (*) is even stronger than our hypothesis and our proof is complete.

We could of course have phrased Theorem 1 more generally in terms of arbitrary Σ-submodels. The three special cases given in [2] can even be shown to hold in an arbitrary admissible set.

3. Roughly speaking, by a functional we will mean a definable mapping from ndexed collections of structures to structures. Is there some very simple sufficient condition under which the functional will preserve "$\equiv_{\infty\omega}$"? Very reasonable conditions have been given in both [3] and [7]. We will suggest some other very simple conditions which, with a trivial proof, will yield a preservation theorem. The three sets of conditions presented in [3], [7] and below appear to be pairwise incomparable with respect to generality. We proceed with some definitions.

It will streamline our notation considerably in this section if by $\mathscr{M} \equiv_{\infty\omega} \mathscr{N}$ we mean that there is some language $\mathscr{L}$ which is the appropriate language for both $\mathscr{M}$ and $\mathscr{N}$ and that $\mathscr{M} \equiv_{\mathscr{L}\infty\omega} \mathscr{N}$. Since the appropriate language can be obtained from the structure in Δ_0 way, no increase in complexity is involved.

DEFINITION 3.1. By a *sequence of structures* we mean a function whose domain is some ordinal and whose range is a set of structures (not necessarily for the same language).

DEFINITION 3.2. A *functional* is a mapping defined on some subclass of sequences of structures which takes on structures as values.

DEFINITION 3.3. A functional F *preserves* $\equiv_{\infty\omega}$ *(isomorphism)* iff whenever i and j are in the domain of F and have some ordinal α as their common domain, and if for each $\beta < \alpha$, $i(\beta) \equiv_{\infty\omega} j(\beta)$ ($i(\beta)$ and $j(\beta)$ are isomorphic), then, $F(i) \equiv_{\infty\omega} F(j)$ ($F(i)$ and $F(j)$ are isomorphic).

We can now state

THEOREM 2. *Suppose F is a functional which*

(i) *preserves isomorphism when restricted to countable sequences of countable* *structures for countable languages in its domain, and which is*

(ii) *Σ-definable with parameters from $H(\omega_1)$.*

Then F preserves $\equiv_{\infty\omega}$.

PROOF. The conclusion of the theorem is that

$$\text{“}\forall i,j,\alpha,\mathcal{M},\mathcal{N}[[\text{domain } i = \text{domain } j \wedge \forall\beta < \alpha(i(\beta) \equiv_{\infty\omega} j(\beta)) \wedge$$

$$F(i) = \mathcal{M} \wedge F(j) = \mathcal{N}] \rightarrow \mathcal{M} \equiv_{\infty\omega} \mathcal{N}]\text{”}$$

holds.

If we write down the above, expressing the first occurrence of "$\equiv$"$_{\infty\omega}$ in a Σ way, i.e., we use fact (II), and the second occurrence in the obvious Π way, then by using (ii) above, we see that we have a Π expression.

If we look at this statement restricted to $H(\omega_1)$, then, by using fact III, we see that we may replace both occurrences of "$\equiv_{\infty\omega}$" by isomorphism. Hypothesis (i) above then assures us that the statement holds on $H(\omega_1)$. An application of (IV), Lévy's theorem, finishes the proof. ⊣

Condition (i) in the hypothesis of the theorem is, of course, no restriction whatsoever. Condition (ii), on the other hand, is a strong restriction. In particular it implies that F does not increase cardinality. Therefore, while one can show that weak direct product will be a Σ functional, full cartesian product cannot be Σ though it does preserve "$\equiv$"$_{\infty\omega}$. Other examples of Σ-functionals include sums and products of linear orderings, taking commutative rings to quotient fields or polynomial rings, taking commutative groups to their subgroups of torsion elements, etc.

Consider all structures with one binary relation satisfying the axiom of extensionality and the functionals taking these structures to their well-founded parts or their non-well-founded parts. These functionals are Σ-definable, so by Theorem 2, if two extensional structures are $\equiv_{\infty\omega}$ then both their well-founded parts and non-well-founded parts are $\equiv_{\infty\omega}$. The second functional does not seem to fit the description of [7]. Neither is it a μ-local functor in the sense of [3] since it does not preserve $\equiv^{\alpha}_{\infty\omega}$ for arbitrary ordinals α.

Before ending this section we must mention that Theorem 2 could be seen from the following even more set-theoretical viewpoint.

Go to a Boolean-valued universe in which everything involved is countable and hence isomorphic. To be sure that (i) is true in this new universe, use Schoenfield's absoluteness lemma (together with facts II and III above). Now, isomorphism in the Boolean extension guarantees '$\equiv$'$_{\infty\omega}$ in the original universe, and we are done.

4. Given a sentence ϕ of $\mathscr{L}_{\infty\omega}$, one of two situations may be true: either ϕ has, up to $\equiv_{\infty\omega}$, a set of models, or ϕ has a proper class of models which are not $\equiv_{\infty\omega}$. In the first case, we say that ϕ *tapers off*. In [8] the following is established.

THEOREM *Let $\mathscr{L}$ have at most μ symbols, where μ is a regular cardinal. Let ϕ be a sentence of $\mathscr{L}_{\mu^+\omega}$. Suppose, up to $\equiv_{\infty\omega}$, there are at most μ models of ϕ of cardinality μ. Then ϕ tapers off. In fact, every model of ϕ is $\equiv_{\infty\omega}$ to a model of cardinality at most μ.*

This result is easily proved using the result of Lévy's. We now state another result in this direction with a weaker hypothesis.

THEOREM 3. *Let μ be a cardinal and let $\alpha < \mu$ be an ordinal. Let $\mathscr{L}$ have at most λ symbols, where $\lambda < \mu$, and let ϕ be a sentence of $\mathscr{L}_{\mu\omega}$. Suppose that for any model $\mathscr{M}$ of ϕ of power less than μ there is a $\beta < \alpha$ such that, for any $\vec{m}, \vec{p} \in \vec{M}$, $\vec{m} \sim_\beta \vec{p}$ implies $\vec{m} \sim_{\beta+1} \vec{p}$. Then ϕ tapers off. In fact, there are, up to $\equiv_{\infty\omega}$, at most $\beth_{\alpha+1}(\lambda)$ models of ϕ, and every model of ϕ is $\equiv_{\infty\omega}$ to a structure of power $\leqq \beth_\alpha(\lambda)$.*

PROOF. As before, we assume that our language $\mathscr{L}$ is an element of $H(\mu)$. Our hypothesis then insures that

$$\langle H(\mu),\varepsilon\rangle \models \text{``}\forall \mathscr{M}[\mathscr{M} \models \phi \to [\exists \beta < \alpha[\forall \vec{m}, \vec{p} \in \mathscr{M}[\vec{m} \sim_\beta \vec{p} \to \vec{m} \sim_{\beta+1} \vec{p}]]]] .$$

It is well known that for each structure $\mathscr{M}$, each k-tuple $\vec{m} \in M$ and ordinal β, there is a formula $\phi^{\beta}_{\vec{m},\mathscr{M}}$ (we usually omit the subscript $\mathscr{M}$), which is Σ in $\mathscr{M}$, $\vec{m}$ and β, such that for every $\vec{p} \in \vec{M}$, $\mathscr{M} \models \phi^{\beta}_{\vec{m}\mathscr{M}}[\vec{p}] \to \vec{m} \sim_\beta \vec{p}$.

It is now clear that above we have a Π statement about $\langle H(\mu),\varepsilon\rangle$, and so, by Levy's theorem, for any model $\mathscr{M}$ of ϕ, there is an ordinal $\beta < \alpha$ such that for all $\vec{m}, \vec{p} \in \vec{M}$, $\vec{m} \sim_\beta \vec{p} \to \vec{m} \sim_{\beta+1} \vec{p}$. It is also well known that for any structure $\mathscr{M}$, the sentence "$\phi^{\beta}_{\phi} \wedge_{k \in \omega\, \vec{m} \in \vec{\lambda} m}[\forall v_1, \cdots, v_k(\phi^{\beta}{}_{\vec{m}} \to \phi^{\beta+1}_{\vec{m}})]$", where

$$\forall\, \vec{m}, \vec{p} \in \vec{M}[\vec{m} \sim_\beta \vec{p} \to \vec{m} \sim_{\beta+1} \vec{p}]$$

is a Scott sentence for $\mathscr{M}$, i.e., characterizes $\mathscr{M}$ up to $\equiv_{\infty\omega}$. We now have merely to count up all the possible Scott sentences of the above form where β is the minimal ordinal such that $\forall\, \vec{m}, \vec{p} \in \vec{M}(\vec{m} \sim_\beta \vec{p} \to \vec{m} \sim_{\beta+1} \vec{p})$.

It can be shown by a straight forward induction argument that for every ordinal γ, structure $\mathscr{M}$, and $\vec{m} \in \vec{M}$, $\phi^{\gamma}_{\vec{m}}$ lies in $H(\beth_{\gamma}(\lambda)^{+})$.

It is then clear from this last observation and our hypothesis that each canonical Scott sentence lies in $H(\beth_{\beta+1}(\lambda)^{+})$ for some $\beta < \alpha$. Adding everything up we find that every canonical Scott sentence is in $\cup_{\delta<\alpha} H(\beth_{\beta+1}(\lambda)^{+})$ and so there are at most $\Sigma_{\beta<\alpha} \beth_{\beta+2}(\lambda) \leqq \beth_{\alpha+1}(\lambda)$ such Scott sentences.

A direct application of the downward Lowenheim-Skolem theorem for $\mathscr{L}_{\infty\omega}$ finishes the proof. ⊣

Though the result given above is in general the best possible, it is obvious that, if α is a limit ordinal, our proof gives a bound of $\beth_{\alpha}(\lambda)$ models. ⊣

As a final note, the analogous result to Theorem 3 for pseudo elementary classes in $\mathscr{L}_{\infty\omega}$ is established exactly as above.

5. To remind the reader that our primary concern is with our method and not our specific results, we outline the essential aspects involved.

First we consider the proposed result from a set-theoretic point of view and analyze the complexity of the statements. Often, purely model theoretic facts come into play at this stage to reduce the complexity.

If we reach the level of formula simplicity we need, we then apply the set theoretical result we have in mind. In this paper we depend entirely on Lévy's theorem. To get other results we could use, for example, theorems of a descriptive set theoretical nature. We might use the fact that Σ-subsets of $H(\omega_1)$ have cardinality less than or equal to ω_1 or equal to the continuum to get Morley's result on the number of countable models of a countable theory. The familiar fact that Borel sets of reals have power continuum or are countable gives us the result that countable structures for countable languages have either continuum many or countably many automorphisms,which was previously shown by model theoretical means. The analogous result for analytic sets allows us to say that a countable theory in $\mathscr{L}_{\omega_1\omega}$ has either continuum or countably many countable homogeneous models up to isomorphism.

As part of the method, especially when we are interested in the number of models up to isomorphism or $\equiv_{\infty\omega}$, etc., we must usually count canonical Scott sentences or some analogous object, instead of the models themselves, so that we do not count equivalent models more then once.

We hope that extensions and refinements of this general approach will prove fruitful in dealing with a variety of questions in "hard" model theory.

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